(eight equations of conservation of mass and momentum of the phases, three transfer equations, and relation (5.1)) for twelve unknowns ($\langle p \rangle$, $\langle \rho \rangle$, the three quantities θ_i, $\psi(\lambda)$, and six velocities). This is one more equation than in the case of monodisperse suspension. However, the system of equations for polydisperse suspension is much more complex than that for a monodisperse suspension, since the equations themselves are integrodifferential.

In conclusion we note that there is generally a size dispersion of particles of equal density; however, in certain applications (ore separation in streams, separation in a pseudoliquefied layer, etc.) it is necessary to consider suspensions dispersed not only over size, but over particle density as well.

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DIFFUSION OF A VORTEX AND CONSERVATION OF MOMENT OF MOMENTUM IN DYNAMICS OF NONPOLAR FLUIDS

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We show that the law of conservation of angular momentum in a flow of an incompressible Stokes fluid can, in a particular case, be reduced to the equation of vortex diffusion. We perform the analysis using two different representations, the Eulerian and the Lagrangian, of the kinetic moment of a fluid particle. We discuss the relevant concepts of the moments of inertia and give an equation for the rate of change of the Lagrangian moment of inertia of a fluid particle.

For the classical (nonpolar) media the law of conservation of the angular momentum can only lead to the condition of symmetry of the stress tensor [1], and nontrivial results can be expected only for the media with microstructure [2]. However when we consider the volumes whose characteristic dimensions are comparable with the scale of the velocity gradient field, then the balance of the angular momentum will necessarily include the kinetic moment and the mean vortical motion. Moreover it appears, that in the case of a nonpolar (e.g. Stokes') fluid, the first terms of the Taylor expansion of the kinetic moment of a particle which are not identically equal to zero, are defined by a vortex motion. We shall show that the kinetic moment of the elementary (from the point of view of the continuum mechanics) volume of the conventional viscous fluid must be taken into account in the study of flows of suspensions containing rotating fluidized particles [3]. This is true particularly in the case of anisotropic turbulent flows [4].

1. Let us consider a volume V of characteristic dimension δ, fixed in some inertial system and bounded by the surface S. Let us write the equation of balance of the angular momentum of an incompressible fluid (of density ρ) contained within the given volume, relative to a fixed point O

$$\int_V \rho \frac{\partial}{\partial t}(\mathbf{M} + \mathbf{R} \times \mathbf{u})\, dV + \int_S \rho(\mathbf{M} + \mathbf{R} \times \mathbf{u}) u_n \, dS = \int_V \rho(\mathbf{G} + \mathbf{R} \times \mathbf{F})\, dV + \int_S (\mathbf{c}_n + \mathbf{R} \times \mathbf{T}_n)\, dS \qquad (1.1)$$

Here $\mathbf{R}$ is the radius vector originating at O, $\mathbf{T}$ is the stress tensor, $\mathbf{M}$ is the internal moment density, $\mathbf{c}$ is the couple stress tensor and $\mathbf{G}$ is the mass momentum density. We set

$$\mathbf{R} = \mathbf{r} + \boldsymbol{\xi} \qquad (1.2)$$

where $\mathbf{r}$ (x_1, x_2, x_3) is the radius vector of the mass center C of the fluid contained within V. Using the Gauss-Ostrogradskii formula to pass from the surface to the volume integrals and introducing the Levi-Civita antisymmetric tensor ε_{ijk}, we can transform (1.1) into

$$\int_V \rho \left\{\frac{\partial M_i}{\partial t} + \frac{\partial M_i u_p}{\partial \xi_p}\right\} dV + \int_V \rho \varepsilon_{ijk} \left\{\frac{\partial R_j u_k}{\partial t} + \frac{\partial}{\partial \xi_p}(R_j u_k u_p)\right\} dV = \int_V \varepsilon_{ijk} \left\{\frac{\partial}{\partial \xi_p}(R_j T_{kp}) + \rho R_j F_k\right\} dV + \int_V \left\{\frac{\partial c_{ip}}{\partial \xi_p} + \rho G_i\right\} dV \qquad (1.3)$$

Using now (1.2), remembering that x_i is constant during the integration with respect to ξ_i and taking into account the condition of incompressibility $\partial u_p / \partial \xi_p = 0$, we obtain

$$\rho \int_V \left(\frac{\partial M_i}{\partial t} + u_p \frac{\partial M_i}{\partial \xi_p}\right) dV + \rho \varepsilon_{ijk} \left\{x_j \int_V \left(\frac{\partial u_k}{\partial t} + u_p \frac{\partial u_k}{\partial \xi_p}\right) dV + \int_V \xi_j \left(\frac{\partial u_k}{\partial t} + u_p \frac{\partial u_k}{\partial \xi_p}\right) dV\right\} = \int_V \left(\frac{\partial c_{ip}}{\partial \xi_p} + \rho G_i\right) dV + \varepsilon_{ijk} \left\{x_j \int_V \left(\frac{\partial T_{kp}}{\partial \xi_p} + \rho F_k\right) dV + \int_V \xi_j \left(\frac{\partial T_{kp}}{\partial \xi_p} + \rho F_k\right) dV + \int_V T_{kj} dV\right\} \qquad (1.4)$$

Let us express the integrands in (1.4) in the form of Taylor series with respect to the mass center C moving with velocity v_k. We shall write the derivatives at the point C (as distinct from the derivatives at the running points inside V) as $\partial v_k / \partial t$ and $\partial v_k / \partial x_l$. We shall have

$$u_i(x_l + \xi_l) = v_i(x_l) + \frac{\partial v_i(x_l)}{\partial x_m} \xi_m + \frac{1}{2} \frac{\partial^2 v_i(x_l)}{\partial x_m \partial x_n} \xi_m \xi_n + o(\delta^2)$$

$$\frac{\partial u_i(x_l + \xi_l)}{\partial \xi_m} = \frac{\partial v_i(x_l)}{\partial x_m} + \frac{\partial^2 v_i(x_l)}{\partial x_m \partial x_n} \xi_n + o(\delta) \qquad (1.5)$$

assuming that

$$\int_V \xi_j dV = 0 \qquad (1.6)$$

As a result, (1.4) takes the form

$$\rho V\left(\frac{\partial M_i}{\partial t}+v_p\frac{\partial M_i}{\partial x_p}\right)+\frac{1}{2}\rho I_{ml}\frac{\partial^2}{\partial x_m\partial x_l}\left(\frac{\partial M_i}{\partial t}+v_p\frac{\partial M_i}{\partial x_p}\right)+$$
$$+\rho\varepsilon_{ijk}\Big\{x_j\Big[V\left(\frac{\partial v_k}{\partial t}+v_p\frac{\partial v_k}{\partial x_p}\right)+\frac{1}{2}I_{ml}\frac{\partial^2}{\partial x_m\partial x_l}\left(\frac{\partial v_k}{\partial t}+v_p\frac{\partial v_k}{\partial x_p}\right)\Big]+$$
$$+I_{jl}\frac{\partial}{\partial x_l}\left(\frac{\partial v_k}{\partial t}+v_p\frac{\partial v_k}{\partial x_p}\right)\Big\}+o(\delta^5)=\varepsilon_{ijk}\Big\{x_j\Big[V\left(\frac{\partial T_{kp}}{\partial x_p}+\rho F_k\right)+$$
$$+\frac{1}{2}I_{ml}\frac{\partial^2}{\partial x_m\partial x_l}\left(\frac{\partial T_{kp}}{\partial x_p}+\rho F_k\right)\Big]+VT_{kj}+\frac{1}{2}I_{ml}\frac{\partial^2 T_{kj}}{\partial x_m\partial x_l}+$$
$$+I_{jl}\frac{\partial}{\partial x_l}\left(\frac{\partial T_{kp}}{\partial x_p}+\rho F_k\right)\Big\}+V\left(\frac{\partial c_{ip}}{\partial x_p}+\rho G_i\right)+$$
$$+\frac{1}{2}I_{ml}\frac{\partial^2}{\partial x_m\partial x_l}\left(\frac{\partial c_{ip}}{\partial x_p}+\rho G_i\right)+o(\delta^5) \qquad (1.7)$$

where ρI_{ml} is the moment of inertia (density) tensor of the material contained within the volume V

$$I_{ml}=\int_V \xi_l\xi_m dV\sim O(\delta^5) \qquad (1.8)$$

Hereafter we adopt for convenience the tensor I^*_{ml}

$$I^*_{ml}=\frac{1}{V\delta^2}I_{ml}\sim O(\delta^\circ) \qquad (1.9)$$

and consider such similar volumes V (δ) with respect to the point C (center of gravity), for which I^*_{ml} is independent of δ and defined only by the form and orientation of V.

Let us rewrite (1.7), collecting all terms of like order in δ

$$\Big\{\rho\left(\frac{\partial M_i}{\partial t}+v_p\frac{\partial M_i}{\partial x_p}-G_i\right)-\varepsilon_{ijk}T_{kj}-\frac{\partial c_{ip}}{\partial x_p}+\varepsilon_{ijk}x_j\Big[\rho\left(\frac{\partial v_k}{\partial t}+v_p\frac{\partial v_k}{\partial x_p}-F_k\right)-$$
$$-\frac{\partial T_{kp}}{\partial x_p}\Big]\Big\}V+\frac{1}{2}I_{ml}\frac{\partial^2}{\partial x_m\partial x_l}\Big\{\rho\left(\frac{\partial M_i}{\partial t}+v_p\frac{\partial M_i}{\partial x_p}-G_i\right)-\varepsilon_{ijk}T_{kj}-\frac{\partial c_{ip}}{\partial x_p}\Big\}+$$
$$+\varepsilon_{ijk}x_jI_{ml}\frac{\partial^2}{\partial x_m\partial x_l}\Big[\rho\left(\frac{\partial v_k}{\partial t}+v_p\frac{\partial v_k}{\partial x_p}-F_k\right)-\frac{\partial T_{kp}}{\partial x_p}\Big]+\varepsilon_{ijk}I_{jl}\frac{\partial}{\partial x_l}\times$$
$$\times\Big\{\rho\left(\frac{\partial v_k}{\partial t}+v_p\frac{\partial v_k}{\partial x_p}-F_k\right)-\frac{\partial T_{kp}}{\partial x_p}\Big\}+o(\delta^5)=0 \qquad (1.10)$$

Here the last group of terms is related to the motion of the fluid in V relative to its center of gravity. The contribution of this motion towards the moment is of the order O (δ^5) and becomes noticeable at sufficiently large V, i.e. when δ are sufficiently near to L, where L is the characteristic linear scale of the gradient field $\partial/\partial x_p$.

Dividing both parts of (1.10) by V, passing to the limit as $\delta\to 0$, i.e. considering (1.10) when $\delta\ll L$, and taking into account the equation of impulses

$$\rho\left(\frac{\partial v_k}{\partial t}+v_p\frac{\partial v_k}{\partial x_p}-F_k\right)-\frac{\partial T_{kp}}{\partial x_p}=0 \qquad (1.11)$$

we obtain the familiar [1, 5] equation of conservation of internal moment

$$\rho\left(\frac{\partial M_i}{\partial t}+v_p\frac{\partial M_i}{\partial x_p}-G_i\right)-\varepsilon_{ijk}T_{kj}-\frac{\partial c_{ip}}{\partial x_p}=0 \qquad (1.12)$$

Below we shall only consider these nonpolar fluids, in which

$$M_i \equiv 0, \qquad G_i \equiv 0, \qquad c_{ip} \equiv 0 \tag{1.13}$$

Under these conditions we find from (1.12), that the stress tensor

$$\varepsilon_{ijk} T_{kj} = 0 \tag{1.14}$$

is symmetric.

Taking into account (1.11) and (1.9), dividing both parts of (1.8) by $\Delta V \delta^2$, and passing to the limit as $\delta \to 0$, we obtain

$$\rho \varepsilon_{ijk} I^*_{jl} \frac{\partial}{\partial x_l} \left(\frac{\partial v_k}{\partial t} + v_p \frac{\partial v_k}{\partial x_p} \right) = \varepsilon_{ijk} \left(I^*_{jl} \frac{\partial^2 T_{kp}}{\partial x_l \partial x_p} + \rho I^*_{jl} \frac{\partial F_k}{\partial x_l} \right) \tag{1.15}$$

which is also the consequence of the impulse equation (1.11).

If $I^*_{jl} = I^* \delta_{jl}$ (which is true e. g. for a spherical or cubic segment of space), then expression (1.15) becomes

$$\rho \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial v_k}{\partial t} + v_p \frac{\partial v_k}{\partial x_p} \right) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial T_{kp}}{\partial x_p} + \rho F_k \right) \tag{1.16}$$

which, in fact, results from taking the curl of (1.9). For an incompressible Stokes'fluid

$$T_{kp} = -p\delta_{kp} + \mu (\partial v_k / \partial x_p + \partial v_p / \partial x_k) \tag{1.17}$$

this becomes the classical equation of vortex diffusion

$$\rho \left(\frac{\partial \omega_i}{\partial t} + v_p \frac{\partial \omega_i}{\partial x_p} \right) = \rho \omega_p \frac{\partial v_i}{\partial x_p} + \mu \frac{\partial^2 \omega_i}{\partial x_p \partial x_p} + \frac{1}{2} \rho \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} \tag{1.18}$$

$$\omega_i = \frac{1}{2} \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$$

2. Let us write the expression for the moment of inertia of the fluid element $V(t)$ present at some instant within the volume V, relative to the center of mass of this element

$$\rho m_i = \rho m^*_i V \delta^2 = \rho \varepsilon_{ijk} \int\limits_{V(t)} \xi_j u_k dV = \rho \varepsilon_{ijk} i_{jl} \frac{\partial v_k}{\partial x_l} + o(\delta^5)$$

$$i_{jl} = \int\limits_{V(t)} \xi_j \xi_l dV \tag{2.1}$$

We now take the particle derivative of m_i, which is equivalent to obtaining an expression for the rate of change of the moment of inertia of the singled out material (remembering that δ denotes the characteristic dimension of the volume V at the initial instant $t = 0$)

$$\rho \frac{dm_i}{dt} = \rho \delta^2 V \frac{dm_i^*}{dt} = \rho \varepsilon_{ijk} \int\limits_{V(t)} \xi_j \frac{du_k}{dt} d\tau, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + u_p \frac{\partial}{\partial \xi_p} \tag{2.2}$$

or

$$\rho \frac{dm_i}{dt} = \delta^2 V \rho \varepsilon_{ijk} i^*_{jl} \frac{\partial}{\partial x_l} \left(\frac{\partial v_k}{\partial t} + v_p \frac{\partial v_k}{\partial x_p} \right) + o(\delta^5) \tag{2.3}$$

$$i^*_{jl}(t) = \frac{1}{V(t)\delta^2} i_{jl}(t), \quad i^*_{jl}(t=0) = I^*_{jl}, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + v_p \frac{\partial}{\partial x_p}$$

in the case of small volumes.

We see that the left-hand side of (1.15) corresponds to the change in the moment of momentum relative to the fixed center of mass. We can similarly show that the first term in the right-hand side of (1.15) corresponds to the moment of the surface forces, and the second term to the moment of mass forces relative to the same fixed center of mass of the particle. We can therefore, following (1.12), bring in c_{ip}^* and g_i^*

$$c_{ip}^* = \varepsilon_{ijk} I_{jl}^* \frac{\partial T_{kp}}{\partial x_l}, \qquad g_i^* = \varepsilon_{ijk} I_{jl}^* \frac{\partial F_k}{\partial x_l} \tag{2.4}$$

and write (1.15) as

$$\rho \left(\frac{dm_i^*}{dt} - g_i^* \right) - \frac{\partial c_{ip}^*}{\partial x_p} = 0 \tag{2.5}$$

We note that for a symmetrical volume $(i_{ij}^* = i\delta_{ij})$ the moment of momentum

$$m_i^* = \varepsilon_{ijk} i_{jl}^* \frac{\partial v_k}{\partial x_l} = i^* \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} = 2i^* \omega_i \tag{2.6}$$

is a function of rotation only (which agrees with the theorem on the kinetic moment of a fluid sphere given in [6]).

3. We now obtain the rate of change of the moment of inertia i_{ij}^* of a particle relative to its moving center of gravity

$$\frac{di^*_{mj}}{dt} = \int\limits_{V(t)} \frac{d}{dt} (\xi_m \xi_j)\, dV$$

Taking into account the fact that (see [2])

$$\frac{d\xi_m}{dt} = u_m - v_m + o(\delta) \tag{3.1}$$

we have

$$\frac{di^*_{mj}(t)}{dt} = \frac{1}{\delta^2 V} \left\{ \int\limits_{V(t)} (u_m - v_m)\, \xi_j\, dV + \int\limits_{V(t)} (u_j - v_j)\, \xi_m\, dV + o(\delta^5) \right\}$$

Following the derivation of (1.5), we expand the integrands into series and pass to the limit as $\delta \to 0$ to obtain

$$\frac{di^*_{mj}}{dt} = i_{lj}^* \frac{dv_m}{\partial x_l} + i_{lm}^* \frac{\partial v_j}{\partial x_l} \tag{3.2}$$

If at the given instant the particle has a symmetrical configuration, then

$$\frac{di^*_{mj}}{dt} = I^* \left(\frac{\partial v_m}{\partial x_j} + \frac{\partial v_j}{\partial x_m} \right) = 2I^* e_{mj} \tag{3.3}$$

i.e. the corresponding instantaneous change of the moment of inertia is caused only by deformation (which may subsequently violate this symmetry). Here e_{mj} denotes the rate of strain component.

Direct differentiation of (2.1) gives

$$\frac{dm_i^*}{dt} = \rho \varepsilon_{ijk} \frac{\partial v_k}{\partial x_l} \frac{di_{jl}^*}{dt} + \rho \varepsilon_{ijk} i_{jl}^* \frac{\partial}{\partial x_l} \left(\frac{\partial v_k}{\partial t} + v_p \frac{\partial v_k}{\partial x_p} \right) - \rho \varepsilon_{ijk} i_{jl}^* \frac{\partial v_p}{\partial x_l} \frac{\partial v_k}{\partial x_p} \tag{3.4}$$

which on comparison with (2.3) yields

$$\rho \varepsilon_{ijk} \frac{\partial v_k}{\partial x_l} \frac{di_{jl}^*}{dt} = \rho \varepsilon_{ijk} i_{jl}^* \frac{\partial v_p}{\partial v_l} \frac{\partial v_k}{\partial x_p} \tag{3.5}$$

The latter can easily be shown to follow from (3.2) and in the case of a symmetrical (at a given instant) particle it simplifies to

$$\rho \varepsilon_{ijk} \frac{\partial v_k}{\partial x_l} \frac{di_{jl}^*}{dt} = \rho I^* \varepsilon_{ijk} \frac{\partial v_p}{\partial x_j} \frac{\partial v_k}{\partial x_p} = \frac{1}{2} \rho I^* \varepsilon_{ijk} \left(\frac{\partial v_k}{\partial x_p} - \frac{\partial v_p}{\partial x_k} \right) \frac{\partial v_p}{\partial x_j} =$$

$$= \rho I^* \varepsilon_{ijk} \frac{\partial v_p}{\partial x_j} \varepsilon_{lpk} \omega_l$$

Using the identity

$$\varepsilon_{ijk}\varepsilon_{lpk} = \delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl} \tag{3.6}$$

we finally obtain the following equation (used in [4])

$$\rho\varepsilon_{ijk}\frac{\partial v_k}{\partial x_l}\frac{di_{jl}^*}{dt} = -\rho I^*\omega_j\frac{\partial v_i}{\partial x_j} \tag{3.7}$$

Thus we see that the first term in the right-hand side of (1.18) is really connected with the change in the moment of inertia of a symmetric particle, the fact already noted by Batchelor [7].

When the volumes of the fluid particles considered are comparable with the cells obtained by sectioning the space by means of the coordinate planes, Eqs. (1.15) and (2.5) take the form of the angular momentum balance equations (in the Eulerian and Lagrangian representations respectively) for a differential volume. In the Cartesian coordinate system, the Eulerian relative moment of inertia of an incompressible fluid contained in a cubical cell $V = dx_1dx_2dx_3$ has the form $I_{ij}^* = {}^1/_{12}\,\delta_{ij}$ and is independent of both, the time and the coordinates.

In the Lagrangian representation the relative moment of inertia i_{jk}^* is associated with the material particle filling the volume $V = dx_1dx_2dx_3$ at the instant t. The variation in i_{ik}^* with the spatial displacement of the particle is caused by its rotation and deformation. It is for this reason that the use of the Lagrangian tensor i_{ij}^* is accompanied by the difficulties associated with the Lagrangian stress tensor.

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